Journal of Approximation Theory **100**, 304–344 (1999)
Article ID jath.1999.3351, available online at http://www.idealibrary.com on IDEAL®

Ratio Asymptotics for Orthogonal Matrix Polynomials*

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Communicated by Walter Van Assche

Received October 24, 1997; accepted in revised form January 15, 1999

Ratio asymptotic results give the asymptotic behaviour of the ratio between two consecutive orthogonal polynomials with respect to a positive measure. In this paper, we obtain ratio asymptotic results for orthogonal matrix polynomials and introduce the matrix analogs of the scalar Chebyshev polynomials of the second kind. © 1999 Academic Press

Key Words: ratio asymptotics; quadrature formulae; orthogonal matrix polynomials; Chebyshev polynomials.

1. INTRODUCTION

In 1979, Nevai established (see [N, Th. 13, p. 33]) the following result concerning ratio asymptotic properties for orthogonal polynomials:

Let $(p_n)_n$ be a sequence of orthonormal polynomials with respect to a measure μ and satisfying the three-term recurrence formula

$$tp_n(t)=a_{n+1}p_{n+1}(t)+b_np_n(t)+a_np_{n-1}(t), \qquad n\geqslant 0,$$

with initial conditions, $p_0(t)=1$ and $p_{-1}(t)=0$ ($(a_n)_n$, $(b_n)_n$ sequences of real numbers with $a_n\neq 0$, $n\geqslant 1$). Assume that

$$\lim_{n\to\infty} a_n=a\neq 0, \qquad \lim_{n\to\infty} b_n=b.$$

Then,

$$\lim_{n\to\infty}\frac{p_{n-1}(z)}{p_n(z)}=\frac{1}{2a}\left((z-b)-\sqrt{(z-b)^2-4a^2}\right),$$

uniformly on compact sets of $\mathbb{C}\setminus\operatorname{supp}(\mu)$.

* This work has been partially supported by DGICYT ref. PB96-1321-C02-00.

0021-9045/99 $30.00

The class of measures satisfying the previous hypothesis is called the Nevai class and it is denoted by $M(a, b)$. Let us notice that

$$\frac{1}{2a}\left((z-b)-\sqrt{(z-b)^2-4a^2}\right)=\frac{1}{2a\pi}\int_{b-2a}^{b+2a}\frac{\sqrt{4a^2-(x-b)^2}\,dx}{z-x}$$

and that

$$\frac{1}{2a^2\pi}\sqrt{4a^2-(x-b)^2}\,\chi_{[b-2a,\,b+2a]}(x)$$

is the weight function for the orthonormal polynomials $(r_n)_n$ which satisfy the following three-term recurrence relation with constant recurrence coefficients:

$$tr_n(t)=ar_{n+1}(t)+br_n(t)+ar_{n-1}(t), \qquad n\geqslant 0.$$

It is clear that $r_n(t)=u_n((t-b)/2a)$, where $(u_n)_n$ are the Chebyshev polynomials of the second kind corresponding to the case $a=\frac{1}{2}$, $b=0$.

Nevai's result completes the so-called Blumenthal theorem, which establishes that for a measure μ in the Nevai class, $\operatorname{supp}(\mu)$ is the compact interval $[b-2a, b+2a]$ and, possibly, two sequences outside this interval which tend to the extreme points of it. Hence, the support of a measure in the Nevai class is bounded. The above ratio asymptotic result has been extended for sequences of orthogonal polynomials in unbounded sets by Van Assche (see [V1, V2]), and recently a technique to find the ratio asymptotics between a polynomial s_n and the nth orthonormal polynomial p_n with respect to a positive measure has been developped by the author (see [D1]).

The purpose of this paper is to extend ratio asymptotics to orthogonal matrix polynomials.

We consider an $N\times N$ positive definite matrix of measures W (for any Borel set $A\subset\mathbb{R}$, $W(A)$ is a positive semidefinite numerical matrix), having moments of every order, i.e., the matrix integral $\int_{\mathbb{R}} t^n\,dW(t)$ exists for any nonnegative integer n.

Assuming that $\int P(t)\,dW(t)\,P^*(t)$ is nonsingular for any matrix polynomial P with nonsingular leading coefficient, the matrix inner product defined in the usual way by W in the space of matrix polynomials has a sequence of orthonormal matrix polynomials $(P_n)_n$, satisfying

$$\int P_n(t)\,dW(t)\,P_m^*(t)=\delta_{n,m}I, \qquad n, m\geqslant 0.$$

$P_n(t)$ is a matrix polynomial of degree n, with a non-singular leading coefficient and is defined up to a multiplication on the left by a unitary matrix.

As in the scalar case, the sequence of orthonormal matrix polynomials $(P_n)_n$ satisfies a three-term recurrence relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \qquad n \geqslant 0, \tag{1.1}$$

where $P_{-1}(t) = \theta$, $P_0(t) \in \mathbb{C}^{N\times N}\backslash\{\theta\}$, A_n are nonsingular matrices and B_n are hermitian. Here and in the rest of this paper, we write θ for the null matrix, the dimension of which can be determined from the context. We remark that the polynomials $Q_n(t) = U_nP_n(t)$, with $U_nU_n^* = I$, are also orthonormal with respect to the same positive definite matrix of measures with respect to which the $(P_n)_n$ are orthonormal, and satisfy a three-term recurrence relation as (1.1) with coefficients $U_{n-1}A_nU_n^*$ instead of A_n and $U_nB_nU_n^*$ instead of B_n.

This three-term recurrence relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures (see, for instance, [AN] or [DL]). In [D2], [D3], and [DV] a very close relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation has been established. This relationship has been used to show that matrix orthogonality is a useful tool for solving certain problems of scalar orthogonality (see [D2, Section 5]). Orthogonal matrix polynomials on the real line are also related to matrix continued fractions and then to network models (see [BB]).

Given two matrices A and B (B hermitian), we say that a sequence of orthonormal matrix polynomials $(P_n)_n$ satisfying (1.1) is in the matrix Nevai class $M(A, B)$ if $\lim_n A_n = A$, $\lim_n B_n = B$. We say that a positive definite matrix of measures W is in the Nevai class $M(A, B)$ if some of its sequences of orthonormal polynomials are in $M(A, B)$. Let us notice that a positive matrix of measures W can belong to several Nevai classes, since the sequence of orthonormal polynomials with respect to W is not unique (recall that orthogonal polynomials are defined up to multiplication on the left by unitary matrices: see above).

When A is nonsingular, we associate to the matrix Nevai class $M(A, B)$ the orthonormal matrix polynomials $(U_n^{A,B})_n$ defined by the recurrence formula

$$tU_n^{A,B}(t) = A^*U_{n+1}^{A,B}(t) + BU_n^{A,B}(t) + AU_{n-1}^{A,B}(t), \qquad n \geqslant 0, \tag{1.2}$$

with initial conditions $U_0^{A,B}(t) = I$, $U_{-1}^{A,B}(t) = \theta$. This sequence is orthonormal with respect to a positive definite matrix of measures we denote by

$W_{A,B}$, which are matrix analogs of the Chebyshev polynomials of the second kind. Let us notice that from $U_0^{A,B}(t)=I$ it follows that $\int dW_{A,B}(t)=I$.

In the scalar case orthonormal polynomials with constant coefficients in the three-term recurrence relation can be reduced to Chebyshev polynomials of the second kind via a linear change of variable (see above). In the matrix case the situation is more interesting and rich. Indeed, we will show that, except for trivial examples (for instance, when A is normal and commutes with B), Chebyshev matrix polynomials $(U_n^{A,B})_n$ form a wide class of essentially different examples of orthonormal matrix polynomials.

To establish the ratio asymptotic results, we need the following definitions: Δ_n stands for the set of zeros of the matrix polynomial P_n, i.e., the zeros of $\det(P_n)$. In [DL], [SV], it is proved that these zeros are real and have multiplicity at most N. We finally put

$$\Gamma=\bigcap_{N\geqslant 0} M_N, \qquad \text{where} \quad M_N=\overline{\bigcup_{n\geqslant N} \Delta_n}. \tag{1.3}$$

It is proved in [DL] that orthogonalizing matrices of measures μ for the matrix polynomials $(P_n)_n$ can be found as weak accumulation points of a sequence of discrete matrices of measures μ_n with support precisely Δ_n. We will show that a matrix of measures W in the Nevai class $M(A,B)$ has compact support and hence is uniquely determined from its sequence of orthogonal matrix polynomials. As a consequence of this we have that $\operatorname{supp}(W)\subset\Gamma$.

In Section 2 we establish the following ratio asymptotic result for the matrix Nevai class $M(A,B)$, assuming that A is nonsingular:

THEOREM 1.1. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation* (1.1). *Assume that* $\lim_n A_n=A$, $\lim_n B_n=B$ *with A nonsingular. Then*

$$\lim_{n\to\infty} P_{n-1}(z)\,P_n^{-1}(z)\,A_n^{-1}=\int\frac{dW_{A,B}(t)}{z-t}, \qquad z\in\mathbb{C}\backslash\Gamma, \tag{1.4}$$

where $W_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the second kind defined by (1.2). *Moveover, the convergence is uniform for z on compact subsets of $\mathbb{C}\backslash\Gamma$.*

Writing

$$F_{A,B}(z)=\int\frac{dW_{A,B}(t)}{z-t}, \tag{1.5}$$

we deduce from this theorem that this analytic matrix function satisfies the matrix equation

$$A^*F_{A,B}(z)\,AF_{A,B}(z)+(B-zI)\,F_{A,B}(z)+I=\theta, \qquad z\in\mathbb{C}\backslash\Gamma. \quad (1.6)$$

We will show that in the matrix version of Nevai's ratio asymptotic result, the matrices P_{n-1} and P_n^{-1} must be multiplied in the order $P_{n-1}P_n^{-1}$; otherwise the result could be false.

We complete Section 2 solving the matrix equation (1.6) when A is hermitian, finding an explicit expression for $F_{A,B}(z)$. To illustrate this expression we show it here for the simpler case when A is positive definite: setting $A^{-1/2}$ for the unique positive definite square root of A we have

$$F_{A,B}(z)=\tfrac{1}{2}A^{-1}(zI-B)\,A^{-1} \\ -\tfrac{1}{2}A^{-1/2}\left(\sqrt{A^{-1/2}(B-zI)\,A^{-1}(B-zI)\,A^{-1/2}-4I}\right)A^{-1/2}. \quad (1.7)$$

The square root which appears in this formula should be understood as follows: if A is positive definite and B is hermitian then we will show that the matrix $A^{-1/2}(zI-B)\,A^{-1/2}$ is diagonalizable except for at most finitely many complex numbers z's, and hence the matrix

$$H_{A,B}(z)=A^{-1/2}(zI-B)\,A^{-1}(zI-B)\,A^{-1/2}-4I \quad (1.8)$$

is also diagonizable, its eigenvalues being of the form a^2-4 where a is an eigenvalue of $A^{-1/2}(zI-B)\,A^{-1/2}$; we take the square root $\sqrt{z}$ such that $|z-\sqrt{z^2-4}|<2$ for $z\in\mathbb{C}\backslash[-2,2]$, and hence the function $z-\sqrt{z^2-4}$ is analytic in $z\in\mathbb{C}\backslash[-2,2]$. We then define the matrix square root

$$\sqrt{A^{-1/2}(zI-B)\,A^{-1}(zI-B)\,A^{-1/2}-4I}$$

in the natural way, i.e., using the diagonal form of the matrix and applying the chosen square root to its eigenvalues.

In Section 3, using the ratio asymptotic results proved in Section 2, we give, when A is positive definite, the following explicit expression for the weight $W_{A,B}$ with respect to which the Chebyshev matrix polynomials $(U_n^{A,B})_n$ are orthonormal: consider the diagonal form of the matrix polynomial $-H_{A,B}(x)$, $x\in\mathbb{R}$, defined by (1.8) (let us notice that for x real, $-H_{A,B}(x)$ is hermitian), that is, $-H_{A,B}(x)=U(x)\,D(x)\,U^*(x)$, where $D(x)$ is a diagonal matrix with entries $d_{i,i}(x)$, $i=1,\ldots,N$, and $U(x)\,U(x)^*=I$. Then the matrix weight $W_{A,B}(x)$, $x\in\mathbb{R}$, has the form

$$dW_{A,B}(x)=\frac{1}{2\pi}\,A^{-1/2}U(x)(D^+(x))^{1/2}\,U^*(x)\,A^{-1/2}\,dx,$$

where $D^+(x)$ is the diagonal matrix with entries

$$d^+_{i,i}(x) = \max\{d_{i,i}(x), 0\}.$$

The support of $W_{A,B}$ is then the set of real numbers

$$\operatorname{supp}(W_{A,B}) = \{x \in \mathbb{R}: A^{-1/2}(xI - B)\, A^{-1/2} \text{ has an eigenvalue in } [-2,2]\}.$$

We will prove that it consists of a finite union of at most N disjoint bounded nondegenerate intervals.

The knowledge of the support of $W_{A,B}$ leads us to extend Blumenthal's theorem for the matrix Nevai class $M(A, B)$ (see [DL] for a partial result in this way). We complete this section with some examples.

It is worth noticing that many interesting questions concerning the weight matrix $W_{A,B}$ for the Chebyshev matrix polynomials of the second kind remain unsolved. Indeed, the main one is to find the general expression for $W_{A,B}$. When A is positive definite, we will prove that $W_{A,B}$ is absolutely continuous with respect to the Lebesgue measure multiplied by the identity matrix, with a continuous matrix Radon–Nikodym derivative and lives on a finite union of at most N disjoint bounded nondegenerate intervals. For A hermitian, we give an example where $W_{A,B}$ is again absolutely continuous with respect to the Lebesgue measure but with an unbounded Radon–Nikodym derivative. But, in the general case, is $W_{A,B}$ still living on at most N disjoint bounded intervals? Are its entries absolutely continuous with respect to the Lebesgue measure, or can Dirac deltas appear?

In Section 4 we study the degenerate case, that is, when A is singular. In a sense we will explain below, this case is surprisingly interesting. We first prove that also for A singular, orthonormal matrix polynomials in $M(A, B)$ have ratio asymptotics:

THEOREM 1.2. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation* (1.1). *Assume that* $\lim_n A_n = A$, $\lim_n B_n = B$ *with A singular. Then there exists a positive definite matrix of measures ν such that*

$$\lim_{n\to\infty} P_{n-1} P_n^{-1}(z)\, A_n^{-1} = \int \frac{d\nu(t)}{z-t} = F_{A,B}(z), \qquad z \in \mathbb{C}\setminus\Gamma,$$

and the convergence is uniform for z on compact subsets of $\mathbb{C}\setminus\Gamma$. Moreover, the analytic function $F_{A,B}$ satisfies that

$$A^* F_{A,B}(z)\, A F_{A,B}(z) + (B - zI)\, F_{A,B}(z) + I = \theta, \qquad z \in \mathbb{C}\setminus\Gamma.$$

Notice that for A singular we have not identified the matrix of measures ν as we did when A is nonsingular (in this case $\nu = W_{A,B}$, where $W_{A,B}$ is the matrix weight of the Chebyshev matrix polynomials). The reason is that the structure of the degenerate case is so rich that it contains many other matrix ratio asymptotic problems corresponding to matrices of lower size. Indeed, consider the N-Jacobi matrix associated to the sequence of matrix Chebyshev polynomials of the second kind satisfying the three-term recurrence relation (1.2), that is, the $(4N-1)$-banded infinite hermitian matrix defined by

$$J = \begin{pmatrix} B & A^* & \theta & \theta & \cdots \\ A & B & A^* & \theta & \cdots \\ \theta & A & B & A^* & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.9}$$

This N-Jacobi matrix can also be considered when A is singular, even though in this case the recurrence formula (1.2) does not define a sequence of matrix polynomials. Then, suitable choices of the singular matrix A will show how many different ratio asymptotic problems are included in this degenerate case. For instance, for the special case when the matrices A and B of size $N \times N$ have the form

$$A = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & b_2 & a_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-2} & b_{N-2} & a_{N-1} \\ 0 & 0 & \cdots & 0 & a_{N-1} & b_{N-1} \end{pmatrix},$$

where $a_0 \neq 0$, the N-Jacobi matrix defined by (1.9) is three diagonal, and it is actually the Jacobi matrix associated to a sequence of scalar orthonormal polynomials: the so-called orthogonal polynomials with periodic recurrence coefficients of period N (studied by, among others, Van Assche and Geronimo, see [V1, Chap. 2], [V3], [GV]). The simplest case of these polynomials comprises the Chebyshev scalar polynomials of the second case which corresponds to the ratio asymptotic problem studied by Nevai. Analogously, we can consider the matrix case of periodic recurrence coefficients with period l. This case is obtained by taking A of size $lN \times lN$ with a unique nonnull block entry of size $N \times N$ in the upper right corner, and B a $(4N-1)$-banded hermitian matrix. The Chebyshev matrix polynomials of the second kind which correspond with the ratio asymptotics problem studied in Theorem 1.1 appear as a particular case.

From what we have explained, it is clear that the identification of the matrix of measures v will be more difficult than for A nonsingular and will have a strong dependence on the particular structure of the matrices A and B. However, we prove that this matrix of measures v is always degenerate, that is, there exists a matrix polynomial P with nonsingular leading coefficient for which $\int P(t)\, dv(t)\, P^*(t)$ is singular (indeed, we show that $\int (tI-B)\, dv(t)(tI-B)^*$ is always singular); consequently, v cannot have a sequence of orthogonal matrix polynomials. We complete this section showing that v can be degenerate in different forms: for instance when $A = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, $a \neq 0$, and $B = \left(\begin{smallmatrix} b & c \\ c & d \end{smallmatrix}\right)$ (which corresponds, as we have explained above, to the orthogonal polynomials with periodic recurrence coefficients of period 2), the matrix of measures v is equal to

$$v = 1 \begin{pmatrix} 1 & \dfrac{t-b}{c} \\ \dfrac{t-b}{c} & \left(\dfrac{t-b}{c}\right)^2 \end{pmatrix} \mu, \tag{1.10}$$

where μ is the positive measure with respect to which the associated orthogonal scalar polynomials with periodic recurrence coefficients are orthonormal.

Another example shows that v can be equal to a matrix whose entries are Dirac deltas on some subspace of $\mathbb{C}^N$ (that is, the case when one eigenvector u of B satisfies $uA^* = \theta$).

2. RATIO ASYMPTOTICS FOR ORTHONORMAL MATRIX POLYNOMIALS

Without loss of generality, we assume $P_0(t) = I$.

In the proof of Theorem 1.1 we need to use that matrix weights in the matrix Nevai class $M_{A,B}$ have compact support:

LEMMA 2.1. *Let $(P_n)_n$ be a sequence of matrix polynomials in the matrix Nevai class $M(A, B)$. Then there exists a positive constant $M > 0$, which does not depend on n, such that $|x_{n,k}| \leqslant M$ for every zero $x_{n,k}$ of P_n. Moreover,* *the sequence* $(P_n)_n$ *has a unique matrix weight* W, *and this matrix weight has compact support contained in* $[-M, M]$. *In particular, the Chebyshev weight* $W_{A,B}$ *has compact support.*

Proof. Consider the N-Jacobi matrix associated to $(P_n)_n$, that is, the $(4N-1)$-banded infinite hermitian matrix defined by

$$J = \begin{pmatrix} B_0 & A_1 & \theta & \theta & \cdots \\ A_1^* & B_1 & A_2 & \theta & \cdots \\ \theta & A_2^* & B_2 & A_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In Lemma 2.1, p. 101 of [DL], it is proved that the zeros of P_n are the eigenvalues of J_{nN} (J_{nN} is the truncated N-Jacobi matrix of dimension nN). Taking into account that $(A_n)_n$ and $(B_n)_n$ converge, and using the Gershgorin disk theorem for the location of eigenvalues, it follows that there exists $M>0$ such that if $x_{n,k}$ is a zero of P_n then $|x_{n,k}| \leqslant M$. So the set of real numbers Γ defined by (1.3) is also contained in $[-M, M]$, and then $\operatorname{supp}(W) \subset \Gamma \subset [-M, M]$. ∎

We are now ready to prove Theorem 1.1, which establishes the ratio asymptotic behaviour of orthonormal matrix polynomials in the matrix Nevai class $M(A, B)$ with A nonsingular. The technique we use in the proof is the matrix version of the proof given in [D1].

Proof. First, we prove that

$$\lim_{n\to\infty} P_{n-1}(z)\, P_n^{-1}(z)\, A_n^{-1} = \int \frac{dW_{A,B}}{z-t},$$

for $z \in \mathbb{C} \setminus \Gamma$.

To do this, we consider the sequence of matrices of discrete measures $(\mu_n)_n$ defined by

$$\mu_n = \sum_{k=1}^{m} \delta_{x_{n,k}} P_{n-1}(x_{n,k})\, \Gamma_{n,k} P_{n-1}^*(x_{n,k}), \qquad n \geqslant 0, \tag{2.1}$$

where $x_{n,k}$, $k=1, ..., m$, are the different zeros of the polynomial P_n, and the matrix $\Gamma_{n,k}$ is given by

$$\Gamma_{n,k} = \frac{1}{(\det(P_n(t))^{(l_k)}(x_{n,k})} (\operatorname{Adj}(P_n(t)))^{(l_k-1)}(x_{n,k})\, Q_n(x_{n,k}), \tag{2.2}$$

l_k being the multiplicity of the zero $x_{n,k}$ and Q_n the polynomial of the second kind. We recall that the matrix $\Gamma_{n,k}$ is the weight in the quadrature formula for $(P_n)_n$ associated to the zero $x_{n,k}$ and that $l_k \leqslant N$ (see

Theorem 3.1, p. 1186 of [D4]). Hence, from this quadrature formula it follows that

$$\int d\mu_n(t) = I, \qquad \text{for} \quad n \geqslant 0.$$

We proceed in several steps:

First step. For a given nonnegative integer n, we have that

$$P_{n-1}(z)\, P_n^{-1}(z)\, A_n^{-1} = \int \frac{d\mu_n(t)}{z-t}. \tag{2.3}$$

In fact, from (3.3), p. 1186 of [D4], it follows that the decomposition

$$P_{n-1}(z)\, P_n^{-1}(z) = \sum_{k=1}^{m} C_{n,k} \frac{1}{z - x_{n,k}}$$

is always possible, even though the zeros of P_n can be of multiplicity greater than one.

It is clear that the matrices $C_{n,k}$ are equal to

$$C_{n,k} = \frac{1}{(\det(P_n(t))^{(l_k)}(x_{n,k})} P_{n-1}(x_{n,k})(\operatorname{Adj}(P_n(t)))^{(l_k-1)}(x_{n,k}).$$

Then we have

$$C_{n,k} A_n^{-1} = \frac{1}{(\det(P_n(t))^{(l_k)}(x_{n,k})} P_{n-1}(x_{n,k})(\operatorname{Adj}(P_n(t)))^{(l_k-1)}(x_{n,k})\, A_n^{-1}.$$

Now, from the Liouville formula for $(P_n)_n$ (see (2.6), p. 1183 of [D4]), it follows that

$$\begin{aligned} C_{n,k} A_n^{-1} = {} & \frac{1}{(\det(P_n(t))^{(l_k)}(x_{n,k})} P_{n-1}(x_{n,k})(\operatorname{Adj}(P_n(t)))^{(l_k-1)}(x_{n,k}) \\ & \times (Q_n(x_{n,k})\, P_{n-1}^*(x_{n,k}) - P_n(x_{n,k})\, Q_{n-1}^*(x_{n,k})). \end{aligned}$$

Part (4) of Theorem 2.3, p. 1184 of [D4] gives that

$$\begin{aligned} C_{n,k} A_n^{-1} = {} & \frac{1}{(\det(P_n(t))^{(l_k)}(x_{n,k})} P_{n-1}(x_{n,k})(\operatorname{Adj}(P_n(t)))^{(l_k-1)}(x_{n,k}) \\ & \times Q_n(x_{n,k})\, P_{n-1}^*(x_{n,k}) \\ = {} & P_{n-1}(x_{n,k})\, \Gamma_{n,k} P_{n-1}^*(x_{n,k}), \end{aligned}$$

and the first step is proved.

Using the first step, we have to prove that

$$\lim_{n\to\infty} \int \frac{d\mu_n(t)}{z-t} = \int \frac{dW_{A,B}(t)}{z-t}, \qquad \text{for} \quad z \in \mathbb{C}\backslash\Gamma.$$

Second step. Let us consider the Chebyshev matrix polynomials of the second kind $(U_n^{A,B})_n$ defined by (1.2). Then

$$\lim_{n\to\infty} \int U_l^{A,B}(t)\, d\mu_n(t) = \begin{cases} I, & \text{for} \quad l=0, \\ \theta, & \text{for} \quad l \neq 0. \end{cases} \tag{2.4}$$

According to the definition of the matrix of measures μ_n, we have

$$\int U_l^{A,B}(t)\, d\mu_n(t) = \sum_{k=1}^{m} U_l^{A,B}(x_{n,k})\, P_{n-1}(x_{n,k})\, \Gamma_{n,k} P_{n-1}^*(x_{n,k}). \tag{2.5}$$

We can write

$$U_l^{A,B}(t)\, P_{n-1}(t) = S_{l,n}(t)\, P_n(t) + \sum_{i=1}^{n} \Delta_{i,l,n} P_{n-i}(t), \tag{2.6}$$

where $S_{l,n}(t)$ is a matrix polynomial of degree not greater than $l-1$ and $\Delta_{i,l,n}$, $i=1, ..., n$, are numerical matrices.

Then, from (2.5) and (2.6), we get

$$\int U_l^{A,B}(t)\, d\mu_n(t) = \sum_{k=1}^{m} \left(S_{l,n}(x_{n,k})\, P_n(x_{n,k}) + \sum_{i=1}^{n} \Delta_{i,l,n} P_{n-i}(x_{n,k}) \right) \times \Gamma_{n,k} P_{n-1}^*(x_{n,k}).$$

The definition of $\Gamma_{n,k}$ and Theorem 2.3(3), p. 1184 of [D4], show that $P_n(x_{n,k})\, \Gamma_{n,k} = \theta$, and so

$$\int U_l^{A,B}(t)\, d\mu_n(t) = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \Delta_{i,l,n} P_{n-i}(x_{n,k}) \right) \Gamma_{n,k} P_{n-1}^*(x_{n,k}). \tag{2.7}$$

Since $2n-i-1 \leqslant 2n-1$ for all $i=1, ..., n$, from the quadrature formula for $(P_n)_n$ (see (3.2) in Theorem 3.1, p. 1186 of [D4]), we get that

$$\int U_l^{A,B}(t)\, d\mu_n(t) = \sum_{i=1}^{n} \int \Delta_{i,l,n} P_{n-i}(t)\, dW(t)\, P_{n-1}^*(t) = \Delta_{1,l,n}. \tag{2.8}$$

Step two will follow if we prove that

$$\lim_{n\to\infty} \Delta_{k,l,n} = \begin{cases} I, & \text{for} \quad k = l+1, \\ \theta, & \text{for} \quad k \neq l+1. \end{cases} \tag{2.9}$$

We use induction on l. When $l=0$ the result is immediate since $\Delta_{1,0,n} = I$ and $\Delta_{i,0,n} = \theta$ for $i \neq 1$.

Now suppose the result is valid up to l. The three-term recurrence formula for the matrix polynomials $(U_n^{A,B})_n$ gives that

$$\begin{aligned} U_{l+1}^{A,B}(t)\, P_{n-1}(t) = {} & (A^{*-1} t U_l^{A,B}(t) - A^{*-1} B U_l^{A,B}(t) \\ & - A^{*-1} A U_{l-1}^{A,B}(t))\, P_{n-1}(t). \end{aligned}$$

The formula (2.6) and the three-term recurrence formula for the matrix polynomials $(P_n)_n$ give that

$$\begin{aligned} \Delta_{k,l+1,n} = {} & A^{*-1}(\Delta_{k,l,n} B_{n-k} + \Delta_{k-1,l,n} A^*_{n-k+1} + \Delta_{k+1,l,n} A_{n-k}) \\ & - A^{*-1} B \Delta_{k,l,n} - A^{*-1} A \Delta_{k,l-1,n}. \end{aligned}$$

For $k \geqslant l+3$ or $k \leqslant l-1$, the induction hypothesis shows that $\lim_{n\to\infty} \Delta_{k,l+1,n} = \theta$. For $k = l, l+1$ and $l+2$, from the induction hypothesis and taking into account that $\lim_n A_n = A$ and $\lim_n B_n = B$, it follows respectively that

$$\lim_{n\to\infty} \Delta_{l,l+1,n} = A^{*-1} A - A^{*-1} A = \theta,$$

$$\lim_{n\to\infty} \Delta_{l+1,l+1,n} = A^{*-1} B - A^{*-1} B = \theta,$$

$$\lim_{n\to\infty} \Delta_{l+2,l+1,n} = A^{*-1} A^* = I.$$

We are now ready to prove that

$$\lim_{n\to\infty} \int \frac{d\mu_n(t)}{z-t} = \int \frac{dW_{A,B}(t)}{z-t}, \qquad \text{for} \quad z \in \mathbb{C} \setminus \Gamma.$$

If not, we can find a complex number $z \in \mathbb{C} \setminus \Gamma$, an increasing sequence of nonnegative integers $(n_m)_m$ and a positive constant C for which

$$\left\| \int \frac{dW_{A,B}(t)}{z-t} - \int \frac{d\mu_{n_m}(t)}{z-t} \right\|_2 \geqslant C > 0, \qquad m \geqslant 0, \tag{2.10}$$

where we write $\|\cdot\|_2$ for the spectral norm of a matrix.

Since $(\mu_n)_n$ is a sequence of positive definite matrices of measures, with compact support contained in $[-M, M]$ (see Lemma 2.1) and $\int d\mu_n = I$, we can obtain (by using Banach–Alaoglu's theorem) a subsequence $(l_m)_m$ from $(n_m)_m$, and a positive definite matrix of measures ν having compact support contained in $[-M, M]$ such that

$$\lim_{m\to\infty} \int f(t)\, d\mu_{l_m}(t) = \int f(t)\, d\nu(t),$$

for any continuous matrix function f defined in $[-M, M]$. Hence, by taking $f(t) = U_l^{A,B}(t)$, we have

$$\lim_{m\to\infty} \int U_l^{A,B}(t)\, d\mu_{l_m}(t) = \int U_l^{A,B}(t)\, d\nu(t).$$

Step two now gives that

$$\int U_l^{A,B}(t)\, d\nu(t) = \begin{cases} I, & \text{for} \quad l = 0, \\ \theta, & \text{for} \quad l \neq 0. \end{cases}$$

But, the sequence of matrix polynomials $(U_l^{A,B})_l$ is orthonormal with respect to $W_{A,B}$, and hence

$$\int U_l^{A,B}(t)\, dW_{A,B}(t) = \begin{cases} I, & \text{for} \quad l = 0, \\ \theta, & \text{for} \quad l \neq 0. \end{cases}$$

Since $(U_l^{A,B})_l$ is a basis of the space of matrix polynomials, and ν, $W_{A,B}$ have compact support, we get that $\nu = W_{A,B}$, and (2.10) is impossible.

The uniform convergence on compact sets of $\mathbb{C}\setminus\Gamma$ follows from the Stieltjes–Vitali theorem, since it is straightforward to see that the entries of the matrix $\int d\mu_n(t)/(z-t)$ are uniformly bounded on compact sets of $\mathbb{C}\setminus\Gamma$. ∎

It is worth noting that the order in which the polynomials P_{n-1} and P_n^{-1} are multiplied in the ratio asymptotic result proved in Theorem 1.1, i.e., $P_{n-1}P_n^{-1}$, is essential to guarantee the validity of this result. Indeed, let W be a positive definite matrix of measures and $(P_n)_n$ its orthonormal matrix polynomials. Let us consider a nonsingular matrix C, and the positive definite matrix of measures defined by $T = CWC^*$. It is clear that $R_n = P_n C^{-1}$ are orthonormal matrix polynomials for T which satisfy the same matrix three-term recurrence relation as the $(P_n)_n$ (but with different initial condition). Hence, if the ratio asymptotic result were true for $P_n^{-1}(z)\,P_{n-1}(z)\,A_n^{-1}$, we would have

$$\lim_n R_n^{-1}(z)\,R_{n-1}(z)=\int\frac{dW_{A,B}(t)}{z-t}\,A,$$

but also

$$\lim_n R_n^{-1}(z)\,R_{n-1}(z)=\lim_n CP_n^{-1}(z)\,P_{n-1}(z)\,C^{-1}=C\int\frac{dW_{A,B}(t)}{z-t}\,AC^{-1},$$

which, in general, is clearly false.

From the three-term recurrence relation for the matrix polynomials $(P_n)_n$ and the above asymptotic result, we can get a matrix formula for the analytic function $F_{A,B}(z)=\int dW_{A,B}(t)(z-t)$. Indeed, multiplying the three-term recurrence formula by $P_n^{-1}(z)$ we find that

$$zI=A_{n+1}P_{n+1}(z)\,P_n^{-1}(z)+B_n+A_n^*P_{n-1}(z)\,P_n^{-1}(z).$$

Taking the limit as $n\to\infty$, we get that

$$zI=F_{A,B}^{-1}(z)+B+A^*F_{A,B}(z)\,A,$$

which can be written as

$$A^*F_{A,B}(z)\,AF_{A,B}(z)+(B-zI)\,F_{A,B}(z)+I=\theta. \tag{2.11}$$

When A is hermitian, we are able to solve this matrix equation, finding an explicit expression for the function $F_{A,B}(z)$. In order to illustrate this procedure we first solve the equation assuming A is positive definite, since in this case the expression for $F_{A,B}(z)$ is simpler.

To show the expression for $F_{A,B}(z)$ we need the following general result for diagonalizable matrices:

LEMMA 2.2. *If A and B are hermitian matrices, then the matrix $Az+B$ is diagonalizable except for at most finitely many complex numbers z's.*

For the sake of completeness we include an elementary proof of this result at the end of this section.

For A positive definite, set $A^{1/2}$ for the unique positive definite square root of A. Applying Lemma 2.2 to $A^{-1/2}(B-zI)\,A^{-1/2}$ (B is hermitian), it follows that the polynomial

$$A^{-1/2}(B-zI)\,A^{-1}(B-zI)\,A^{-1/2}-4I \tag{2.12}$$

is diagonalizable except for at most finitely many complex numbers z's (notice that these exceptional values can not be real numbers since (2.12) is hermitian for z real). Its eigenvalues are of the form a^2-4, for a an eigenvalue of $A^{-1/2}(B-zI)\,A^{-1/2}$. If we take the square root $\sqrt{z}$ such that $|z-\sqrt{z^2-4}|<2$ for $z\in\mathbb{C}\setminus[-2,2]$, then the function $z-\sqrt{z^2-4}$ is analytic in $z\in\mathbb{C}\setminus[-2,2]$. For z such that the eigenvalues of $A^{-1/2}(B-zI)\,A^{-1/2}$ are in $\mathbb{C}\setminus[-2,2]$, we define the matrix

$$\sqrt{A^{-1/2}(zI-B)\,A^{-1}(zI-B)\,A^{-1/2}-4I}$$

in the natural way, i.e., using the diagonal form of the matrix and applying the chosen square root to its eigenvalues.

First of all, we remark that this definition does not depend on the choice of the diagonal form of $A^{-1/2}(B-zI)\,A^{-1/2}$ (see for instance [HJ2, pp. 407–408]). The chosen matrix square root defines an analytic function of z in $\mathbb{C}\setminus\Delta$, where

$$\Delta=\{z\in\mathbb{C}: A^{-1/2}(B-zI)\,A^{-1/2} \text{ has at least one eigenvalue in } [-2,2]\},$$

which is a set of real numbers.

Indeed, the Cauchy formula

$$\frac{1}{2\pi i}\int_{\Gamma_z}\sqrt{t^2-4}\,(tI-A^{-1/2}(B-zI)\,A^{-1/2})^{-1}\,dt$$

defines a matrix analytic function of z in $\mathbb{C}\setminus\Delta$, where Γ_z is any simple closed rectifiable curve, contained in $\mathbb{C}\setminus[-2,2]$, that strictly encloses all of the eigenvalues of $A^{-1/2}(B-zI)\,A^{-1/2}$ (the square root $\sqrt{t^2-4}$ is taken as before). Using the diagonal form of $A^{-1/2}(B-zI)\,A^{-1/2}$ it is easy to see that the above analytic function coincides with the definition of the square root we have taken. The set Δ is included in $\mathbb{R}$ because for $z\in\mathbb{C}\setminus\mathbb{R}$, the eigenvalues of $A^{-1/2}(B-zI)\,A^{-1/2}$ are always in $\mathbb{C}\setminus\mathbb{R}$ (if not there exists a nonnull vector u for which $uA^{-1/2}(B-zI)\,A^{-1/2}u^*$ is real; since $A^{-1/2}$ and B are hermitian, this implies that $\mathcal{I}(z)\,uA^{-1}u^*=0$, which is impossible when $\mathcal{I}(z)\neq 0$ because A is nonsingular). We thus conclude that the matrix function

$$A^{-1/2}(B-zI)\,A^{-1/2}-\sqrt{A^{-1/2}(zI-B)\,A^{-1}(zI-B)\,A^{-1/2}-4I}$$

is analytic in $\mathbb{C}\setminus\Delta$, where $\Delta\subset\mathbb{R}$, and $x\in\Delta$ if and only if $A^{-1/2}(B-zI)\,A^{-1/2}$ has at least one eigenvalue in $[-2,2]$.

With this definition, we have the following expression for the function $\int dW_{A,B}(t)/(z-t)$:

COROLLARY 2.3. *Let A be positive definite and B hermitian. Then*

$$\int \frac{dW_{A,B}(t)}{z-t} = \frac{1}{2} A^{-1}(zI-B) A^{-1} - \frac{1}{2} A^{-1/2}\left(\sqrt{A^{-1/2}(B-zI) A^{-1}(B-zI) A^{-1/2}-4I}\right) A^{-1/2}, \tag{2.13}$$

when $z \notin \operatorname{supp}(W_{A,B})$.

Proof. The starting point is the matrix equation for the matrix function $F_{A,B}$:

$$A^* F_{A,B}(z) A F_{A,B}(z) + (B-zI) F_{A,B}(z) + I = \theta.$$

If we write $G_{A,B}(z) = A^{1/2} F_{A,B}(z) A^{1/2}$, we get for $G_{A,B}(z)$ the equation

$$G^2_{A,B}(z) + A^{-1/2}(B-zI) A^{-1/2} G_{A,B}(z) + I = \theta. \tag{2.14}$$

From Lemma 1.1 we can take $M>0$ such that $\operatorname{supp}(W_{A,B}) \subset [-M, M]$. For a given real number $z>M$, we have that $F_{A,B}(z)$ is positive semi-definite since $W_{A,B}$ is a positive definite matrix of measures and $z-t>0$ for $t \in \operatorname{supp}(W_{A,B})$, and so is $G_{A,B}(z)$ because A is positive definite. Since $A^{-1/2}(B-zI) A^{-1}(B-zI) A^{-1/2}-4I$ is positive definite for z large enough, it follows that, for z positive and large enough, the solution of the equation (2.14) equal to $G_{A,B}(z)$ can only be of the form

$$G_{A,B}(z) = \tfrac{1}{2} A^{-1/2}(zI-B) A^{-1/2} - T,$$

where the T is a square root of the matrix $A^{-1/2}(B-zI) A^{-1}(B-zI) A^{-1/2}-4I$ defined using the diagonal form and applying any choice of the square root of the eigenvalues. Since $\lim_{z\to\infty} F_{A,B}(z) = \theta$, we have that

$$G_{A,B}(z) = \tfrac{1}{2} A^{-1/2}(zI-B) A^{-1/2} - \tfrac{1}{2}\sqrt{A^{-1/2}(B-zI) A^{-1}(B-zI) A^{-1/2}-4I},$$

for z positive and large enough, where the square root of the matrix is taken as it was explained above.

Hence we get that

$$\int \frac{dW_{A,B}(t)}{z-t} = \frac{1}{2} A^{-1}(zI-B) A^{-1} - \frac{1}{2} A^{-1/2}\left(\sqrt{A^{-1/2}(B-zI) A^{-1}(B-zI) A^{-1/2}-4I}\right) A^{-1/2},$$

for z positive and large enough. But the expressions on both sides of the previous equality are not analytic in $\operatorname{supp}(W_{A,B})$ and Δ, respectively. Hence, we deduce that $\Delta = \operatorname{supp}(W_{A,B})$, and the corollary is proved. ∎

Before going to the hermitian case, we give an example:

EXAMPLE 1. We take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A straightforward computation gives that

$$A^{-1/2}(B - zI)\, A^{-1}(B - zI)\, A^{-1/2} - 4I = \begin{pmatrix} z^2 & -10z \\ -10z & 16z^2 \end{pmatrix}.$$

This matrix has the following system of eigenvalues and eigenvectors:

$$\frac{17z^2 - 5\sqrt{16z^2 + 9z^4}}{2} \left(\frac{3z}{4} + \frac{\sqrt{16z^2 + 9z^4}}{4z}, 1 \right),$$

$$\frac{17z^2 + 5\sqrt{16z^2 + 9z^4}}{2} \left(\frac{3z}{4} - \frac{\sqrt{16z^2 + 9z^4}}{4z}, 1 \right).$$

According to our choice for the matrix square root, we have

$$\int \frac{dW_{A,B}(t)}{z - t} = \frac{1}{2} \begin{pmatrix} z & -4 \\ -4 & 16z \end{pmatrix}$$

$$- \frac{1}{2\left(\sqrt{17z^2 - 5\sqrt{16z^2 + 9z^4}} + \sqrt{17z^2 + 5\sqrt{16z^2 + 9z^4}}\right)}$$

$$\times \begin{pmatrix} \sqrt{2}\, z^2 + \sqrt{32z^4 - 200z^2} & -20\sqrt{2}\, z \\ -20\sqrt{2}\, z & 64\sqrt{2}\, z^2 + 4\sqrt{32z^4 - 200z^2} \end{pmatrix}.$$

where the square roots are chosen such that $F_{A,B}$ is analytic in $\mathbb{C} \setminus [-\frac{5}{2}, \frac{5}{2}]$.

When A is hermitian we proceed as follows. Multiplying (2.11) to the right by A, we find, taking into account that A is hermitian, that

$$(B - xI)\, F_{A,B}(x)\, A = A F_{A,B}(x)(B - xI), \qquad x \in \mathbb{R}. \tag{2.15}$$

Multiplying (2.11) to the right by $(B-xI)$ and using (2.15), we get the following equation for $F_{A,B}(x)$, $x\in\mathbb{R}$:

$$AF_{A,B}(x)(B-xI)\,F_{A,B}(x)\,A+(B-xI)\,F_{A,B}(x)(B-xI)+(B-xI)=\theta. \tag{2.16}$$

Write $b_1\leqslant b_2\leqslant\cdots\leqslant b_N$ for the eigenvalues of B (they are real because B is hermitian). It is clear that for $x<b_1$ the matrix $(B-xI)$ is positive definite. We now solve (2.16) as in the case when A is positive definite (see Corollary 2.3). Indeed, set $(B-xI)^{1/2}$ for the unique positive definite square root of $(B-xI)$, $x<b_1$, $K_{A,B}(x)=(B-xI)^{1/2}\,F_{A,B}(x)(B-xI)^{1/2}$ and $T_{A,B}(x)=(B-xI)^{1/2}\,A^{-1}(B-xI)^{1/2}$. Then, we find for $K_{A,B}(x)$ the equation

$$K^2_{A,B}(x)+T^2_{A,B}(x)\,K_{A,B}(x)+T^2_{A,B}(x)=\theta, \qquad x<b_1, \tag{2.17}$$

from which we get that

$$\begin{aligned}F_{A,B}(x)=\tfrac{1}{2}A^{-1}(zI-B)\,A^{-1}-\tfrac{1}{2}A^{-1}(B-xI)^{1/2}&\\ \times[I-4(B-xI)^{-1/2}\,A(B-xI)^{-1}\,A(B-xI)^{-1/2}]^{1/2}&\\ \times(B-xI)^{1/2}\,A^{-1},&\end{aligned} \tag{2.18}$$

when $x<b_1$ with $|x|$ large enough so that $1-4(B-xI)^{-1/2}\,A(B-xI)^{-1}$ $A(B-xI)^{-1/2}$ is positive definite.

We now explain how to get an analytic continuation for the formula (2.18). We take the square root such that $\sqrt{(b_i-z)(b_j-z)}$, $i, j=1, ..., N$, is analytic in $\mathbb{C}\setminus[b_1, b_N]$ and positive for positive numbers. With this choice, for a given analytic matrix function $L(z)$ in $\Omega\subset\mathbb{C}$, we can define the product $(B-zI)^{1/2}\,L(z)(B-zI)^{1/2}$, or respectively $(B-zI)^{-1/2}\,L(z)$ $(B-zI)^{-1/2}$, to be analytic at least in $\Omega\setminus[b_1, b_N]$ (this is because the entries of both matrix products are linear combinations of products of the entries of L and $((b_i-z)(b_j-z))^{1/2}$, or $((b_i-z)(b_j-z))^{-1/2}$, respectively). All the matrix products of this type which appear in the formula (2.18) are defined in this way. We now explain how to take

$$\sqrt{I-4(B-zI)^{-1/2}\,A(B-zI)^{-1}\,A(B-zI)^{-1/2}}.$$

To do this we need the following lemma, which will be proved at the end of this section

Lemma 2.4. *If A and B are hermitian matrices, then the matrix $(B-zI)^{1/2}\,A^{-1}(B-zI)^{1/2}$ is diagonalizable except for at most finitely many complex numbers in $\mathbb{C}\setminus[b_1, b_N]$.*

According to this lemma the matrix

$$1-4(B-zI)^{-1/2}A(B-zI)^{-1}A(B-zI)^{-1/2}$$

is diagonalizable when $z\in\mathbb{C}\setminus[b_1,b_N]$ except for at most finitely many complex numbers, and its eigenvalues are of the form $1-4a^{-2}$, for a an eigenvalue of $(B-zI)^{1/2}A^{-1}(B-zI)^{1/2}$. Taking the square root such that $|z-\sqrt{z^2-4}|<2$ for $z\in\mathbb{C}\setminus[-2,2]$ and when $(B-zI)^{1/2}A^{-1}(B-zI)^{1/2}$ has all its eigenvalues in $\mathbb{C}\setminus[-2,2]$, we can define

$$\sqrt{1-4(B-zI)^{-1/2}A(B-zI)^{-1}A(B-zI)^{-1/2}},$$

in the natural way (as we did for the case A positive definite), i.e., using the diagonal form and applying the chosen square root to its eigenvalues.

This shows that the formula (2.16) defines an analytic function at least in $\mathbb{C}\setminus(\Delta\cup[b_1,b_N])\}$, where

$$\Delta=\{x\in\mathbb{C}\setminus[b_1,b_N]\colon (B-zI)^{1/2}A^{-1}(B-zI)^{1/2} \text{ has at east one eigenvalue in } [-2,2]\}\subset\mathbb{R}. \tag{2.19}$$

Under additional hypothesis on A and B, the formula (2.16) can still be analytic in a bigger set than $\mathbb{C}\setminus\{\Delta\cup[b_1,b_N]\}$, because the multiplication between the different roots which appear in the formula could extend the analyticity of the product to some subintervals of $[b_1,b_N]$. To illustrate this fact, we give some examples:

Example 2. In the first example, we compare the formula (2.16) with the one we gave in Corollary 2.3 assuming that A is positive definite.

To do this, consider the set of real numbers defined by (2.17) and write Δ_1 for

$$\Delta_1=\{x\colon A^{-1/2}(B-zI)A^{-1/2} \text{ has one eigenvalue in } [-2,2]\}\subset\mathbb{R}.$$

For $x\in\mathbb{R}\setminus[b_1,b_N]$ we define the square root $(B-xI)^{1/2}$ of $(B-xI)$ in the following way: (1) for $x<b_1$, since $(B-xI)$ is positive definite, we take $(B-xI)^{1/2}$ as its unique positive definite square root; (2) for $b_N<x$, since $(B-xI)$ is negative definite, we take $(B-xI)^{1/2}$ as its unique square root of the form iT with T positive definite. Then, for $x\in\mathbb{R}\setminus[b_1,b_N]$ we have that

$$A^{-1/2}(B-xI)A^{-1/2}=((B-xI)^{1/2}A^{-1/2})^*((B-xI)^{1/2}A^{-1/2})$$

$$(B-xI)^{1/2}A^{-1}(B-xI)^{1/2}=((B-xI)^{1/2}A^{-1/2})((B-xI)^{1/2}A^{-1/2})^*,$$

which gives that for $x \in \mathbb{R} \setminus [b_1, b_N]$ the matrices $A^{-1/2}(B-xI)\, A^{-1/2}$ and $(B-xI)^{1/2}\, A^{-1}(B-xI)^{1/2}$ have the same eigenvalues and so $\Delta \setminus [b_1, b_N] = \Delta_1 \setminus [b_1, b_N]$. A simple computation also shows that both formulas for $F_{A,B}$ ((2.13) and (2.18)) define the same function, and then, the formula (2.18) is, in this case, also analytic for $z \in [b_1, b_N] \setminus \Delta_1$, which in general is not empty, as the following example shows: Take

$$A^{-1/2} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

This gives

$$A^{-1/2}(B-xI)\, A^{-1}(B-xI)\, A^{-1/2} - 4I = \begin{pmatrix} x^2+4 & -6\sqrt{2}\,x \\ -6\sqrt{2}\,x & 4x^2+4 \end{pmatrix},$$

which is not positive definite in

$$\Delta_1 = \left[-\sqrt{\frac{13+\sqrt{153}}{2}}, -\sqrt{\frac{13-\sqrt{153}}{2}} \right] \cup \left[\sqrt{\frac{13-\sqrt{153}}{2}}, \sqrt{\frac{13+\sqrt{153}}{2}} \right],$$

and it is clear that

$$[-2, 2] \setminus \Delta_1 = \left[-\sqrt{\frac{13-\sqrt{153}}{2}}, \sqrt{\frac{13+\sqrt{153}}{2}} \right] \neq \varnothing.$$

EXAMPLE 3. Assume that $(B-zI)^{-1/2}\, A(B-zI)^{-1}\, A(B-zI)^{-1/2}$ is diagonal. In this case the analyticity of $F_{A,B}$ can be extended to those points x of the interval $[b_1, b_N]$ for which $(B-xI)^{1/2}\, A^{-1}(B-xI)^{1/2}$ has its eigenvalues in $\mathbb{C} \setminus [-2, 2]$. We illustrate this with the following example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then, we have

$$(B-zI)^{-1/2}\, A(B-zI)^{-1}\, A(B-zI)^{-1/2} = \frac{1}{z^2-2z}\, I.$$

and the evaluation of (2.18) gives that

$$F_{A,B}(z) = -\frac{1}{2}\left[\begin{pmatrix} 1-z & -1 \\ -1 & 1-z \end{pmatrix} + \sqrt{1-\frac{4}{z^2-2z}} \begin{pmatrix} z-1 & 1 \\ 1 & z-1 \end{pmatrix} \right],$$

which is analytic in $\mathbb{C}\backslash\{[1-\sqrt{5},0]\cup[2,1+\sqrt{5}]\}$. Notice that only for $x\in[1-\sqrt{5},0]\cup[2,1+\sqrt{5}]$ one of the eigenvalues of the matrix

$$(B-xI)^{1/2}A^{-1}(B-xI)^{1/2}=\sqrt{x^2-2x}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

is in $[-2,2]$, and that $F_{A,B}(z)$ is in this case analytic on the whole interval $[b_1,b_2]=[0,2]$.

We have solved the matrix Eq. (2.11) for A hermitian and in a simpler way for A positive definite, hence one can ask if for $(P_n)_n$ in the matrix Nevai class $M(A,B)$, there is a sequence of unitary matrices $(V_n)_n$ in such a way that the sequence $(V_nP_n)_n$ is in the Nevai class $M(A',B')$ with A' hermitian, or even better positive definite. The answer is no. First we prove that it is not possible to force A' to be positive definite. Indeed, let us consider the Chebyshev matrix polynomials $(U_n^{A,B})_n$ where

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

The matrix coefficients in the three-term recurrence relation for $(V_nU_n^{A,B})_n$ are given by $V_{n-1}AV_n^*$, $V_nBV_n^*$. First, it is easy to see that the matrix $V_{n-1}AV_n^*$ is unitary and so, it is normal. Hence it is unitary diagonalizable and its eigenvalues are 1 or -1. If the sequence $(V_{n-1}AV_n^*)_n$ converges to a positive definite matrix, we deduce that its eigenvalues converge to the eigenvalues of this positive definite matrix; hence, for n big enough the eigenvalues of $V_{n-1}AV_n^*$ must be equal to 1; i.e., for n big enough we get $V_{n-1}AV_n^*=I$, which gives $V_n=V_{n-1}A$. It is now straightforward to check that the sequence $(V_nB_nV_n^*)_n$ cannot be convergent because $B\neq ABA$.

Proceeding in a similar way with the Chebyshev matrix polynomials $(U_n^{A,B})_n$ where now

$$A=\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix},\qquad B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

it can be proved that for any sequence $(V_n)_n$ of unitary matrices, for which $(V_{n-1}AV_n^*)_n$ converges to a hermitian matrix, the sequence $(V_nBV_n^*)_n$ does not converge.

To finish this section we prove Lemmas 2.2 and 2.3. To do this we use the following result:

LEMMA 2.5. *Let $a_N(z), ..., a_0(z)$ be polynomials with complex coefficients and $a_N(z)$ not identically zero, and consider the polynomial*

$$p_z(t) = a_N(z)\, t^N + a_{N-1}(z)\, t^{N-1} + a_1(z)\, t + a_0(z). \tag{2.20}$$

Then

(1) *except for finitely many z's the polynomial $p_z(t)$ has the same number of different roots.*

(2) *if we denote these roots by $\alpha_1(z), ..., \alpha_m(z)$ then the coefficients of the polynomial*

$$q_z(t) = (t - \alpha_1(z)) \cdots (t - \alpha_m(z)),$$

are rational functions in the variable z.

Proof. The lemma is a consequence of the following fact: given two polynomials $r_z(t)$, $s_z(t)$ of the form (2.20) with $\mathrm{dgr}(r_z) \geqslant \mathrm{dgr}(s_z)$, either (a) $r_z(t)$, $s_z(t)$ have a common root only for finitely many z's; (b) $r_z(t)$ can be decomposed in a product of at least two polynomials of the form (2.20) with the property that any two of these factors have a common root only for finitely many z's; or, finally, (c) $r_z(t) = a(z)(t - b(z))^m$ where $a(z)$ and $b(z)$ are polynomials in the variable z.

Indeed, $r_z(t)$ and $s_z(t)$ have a common root if and only if the resultant $R(r_z, s_z) = 0$ (see [J, p. 309]). But the resultant $R(r_z, s_z)$ is a polynomial in the variable z, so we deduce that if $R(r_z, s_z)$ is not identically zero, then $r_z(t)$ and $s_z(t)$ have a common root only for finitely many z's (the roots of $R(r_z, s_z)$). Otherwise, $R(r_z, s_z)$ is identically zero and then for all z $r_z(t)$ and $s_z(t)$ have a common root. If this happens then, by applying Euclid's algorithm to $r_z(t) = s_z(t) = 0$, we can find a polynomial $b(z)$ (not identically zero) such that $b(z)\, r_z(t) = u_z(t)\, v_z(t)$ where $u_z(t)$ and $v_z(t)$ are polynomials of the form (2.20) with degree greater than 0. Then, the result follows by applying the same procedure to u_z and v_z.

To prove the lemma we proceed by induction on N. The result is trivial for $N = 1$. Assume now the result for N, and take a polynomial $p_z(t)$ of degree $N + 1$ whose coefficients are polynomials in the variable z. Since *$p_z(t)$ has a double root* and *$p_z(t)$, $p'_z(t)$ have a common root* are equivalent properties, we deduce from the result proved above that, either (a) $p_z(t)$ has simple roots except for finitely many z's, or (b) $p_z(t)$ can be decomposed in a product of at least two polynomials of the form (2.20) with the property that any two of these factors have a common root only for finitely many z's, or, finally, (c) $p_z(t) = a(z)(t - b(z))^m$ where $a(z)$ and $b(z)$ are polynomials in the variable z. From (a) and (c) the proof follows directly. From (b) the proof follows straighforwardly by applying the induction hypothesis. ∎

Now, we prove Lemma 2.2: Given A and B hermitian matrices, let $p_z(t)$ be the characteristic polynomial of $Az+B$,

$$p_z(t)=\det(tI-Az-B)=t^N+a_{N-1}(z)\,t^{N-1}+a_1(z)\,t+a_0(z),$$

where the $a_{N-1}(z)$, ..., $a_0(z)$ are polynomials with real coefficients. We now apply the previous lemma to $p_z(t)$ and find the polynomial

$$q_z(t)=(t-\alpha_1(z))\cdots(t-\alpha_m(z)),$$

whose coefficients are rational functions in the variable z and where, except for at most finitely many z's, $\alpha_1(z)$, ..., $\alpha_m(z)$ are the different eigenvalues of $Az+B$. From this it follows that the matrix function $F(z)=q_z(Az+B)$ is analytic in $\mathbb{C}\backslash\Delta$, where Δ is a finite set of complex numbers.

According to the characterization of diagonalizable matrices (see [HJ1, Corollary 3.3.8, p. 145]), $Az+B$ is diagonalizable if and only if $q_z(Az+B)=\theta$. Since for x real the matrix $Ax+B$ is diagonalizable (it is hermitian), we have that $F(x)=\theta$, for $x\in\mathbb{R}\backslash\Delta$ and then, since $F(z)$ is analytic in $\mathbb{C}\backslash\Delta$, it follows $F(z)=\theta$, for $z\in\mathbb{C}\backslash\Delta$, that is, $Az+B$ is diagonalizable except for at most finitely many z's. ∎

It is worth noting that there are indeed points such that $Az+B$ is not diagonalizable. For instance, if we take

$$A=\begin{pmatrix}2 & i\\ -i & 2\end{pmatrix},\qquad B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

and $z=i$, we have that the matrix $Ai+B$ is not diagonalizable.

The proof of Lemma 2.3 is similar, because the characteristic polynomial of $(B-zI)^{1/2}\,A(B-zI)^{1/2}$ is also of the form (2.20).

3. CHEBYSHEV MATRIX POLYNOMIALS OF THE SECOND KIND

In this section, we will give an explicit expression for the matrix of measures $W_{A,B}$, when A is positive definite. To do this, let us consider the matrix polynomial under the square root in the formula (2.13) given in Corollary 2.3 for the Hilbert transform of $W_{A,B}$, that is,

$$H_{A,B}(z)=A^{-1/2}(B-zI)\,A^{-1}(B-zI)\,A^{-1/2}-4I.$$

From Lemma 2.2 we have that $-H_{A,B}(z)$ is diagonalizable except for at most finitely many complex numbers z's, hence we can diagonalize it in the form $-H_{A,B}(z)=U(z)\,D(z)\,U^{-1}(z)$, where $D(z)$ is a diagonal matrix with entries $d_{i,i}(z)$, $i=1, ..., N$.

Since for x real $-H_{A,B}(x)$ is hermitian, it is always diagonalizable with $U(x)\,U(x)^*=I$ and $d_{i,i}(x)$ is real for $i=1, ..., N$. We will prove that the matrix of functions $W_{A,B}(x)$, $x\in\mathbb{R}$, has the form

$$dW_{A,B}(x)=\frac{1}{2\pi}\,A^{-1/2}U(x)(D^+(x))^{1/2}\,U^*(x)\,A^{-1/2}dx, \tag{3.1}$$

where $D^+(x)$ is the diagonal matrix with entries $d^+_{i,i}(x)=\max\{d_{i,i}(x),0\}$. In the formula (3.1), we take the positive square root of the positive entries of the diagonal matrix $D^+(x)$.

First, we remark that the definition of $W_{A,B}$ does not depend on the choice of the diagonal form of $H_{A,B}$, as it can easily be proved (see also [HJ2, p. 407–408]).

From its definition it is clear that $W_{A,B}$ is a positive definite matrix of measures and that when $-H_{A,B}(x)$ is positive definite or semidefinite,

$$dW_{A,B}(x)=\frac{1}{2\pi}\,A^{-1/2}(4I-A^{-1/2}(B-xI)\,A^{-1}(B-xI)\,A^{-1/2})^{1/2}\,A^{-1/2}\,dx.$$

In the following theorem we prove that the matrix of measures $W_{A,B}$ has, indeed, the form given by (3.1) and show other important properties for $W_{A,B}$:

THEOREM 3.1. *If A is positive definite and B hermitian, the matrix weight $W_{A,B}$ for the Chebyshev matrix polynomials of the second kind defined by* (1.2) *is the matrix of measures given by* (3.1). *$W_{A,B}$ is absolutely continuous with respect to the Lebesgue measure multiplied by the identity matrix, with a continuous matrix Radon–Nikodym derivative and lives on a finite union of at most N disjoint bounded nondegenerate intervals whose extreme points are some roots of the scalar polynomial*

$$\det(A^{-1/2}(B-zI)\,A^{-1}(B-zI)\,A^{-1/2}-4I).$$

Proof. From the inversion formula for the Hilbert transform it is enough to prove that

$$\frac{1}{2\pi}\,A^{-1/2}U(x)(D^+(x))^{1/2}\,U^*(x)\,A^{-1/2}$$

$$=-\frac{1}{2\pi i}\lim_{y\to 0}\,(F_{A,B}(x+iy)-F_{A,B}(x-iy)), \qquad x\in\mathbb{R}.$$

For a fixed real number x, we have that $-H_{A,B}(x)$ is hermitian. Hence according to [R, Theorem 1, pp. 33–34] there exist analytic matrices $U(z)$, $D(z)$, at x such that, for z in a neighbourhood of x, $U(z)$ is nonsingular, $U(x)$ is unitary, $D(z)$ is diagonal and

$$-H_{A,B}(z)=U(z)\,D(z)\,U^{-1}(z).$$

We then write

$$F_{A,B}(z)=\tfrac{1}{2}A^{-1}(zI-B)\,A^{-1}-\tfrac{1}{2}A^{-1/2}U(z)\sqrt{-D(z)}\,U^{-1}(z)\,A^{-1/2},$$

where we take the square root as explained in the Introduction.

Then we have

$$\lim_{y\to 0^+}(F_{A,B}(x+iy)-F_{A,B}(x-iy))$$

$$=-\frac{1}{2}\lim_{y\to 0}A^{-1/2}(U(x+iy)\sqrt{-D(x+iy)}\,U^{-1}(x+iy)$$

$$-U(x-iy)\sqrt{-D(x-iy)}\,U^{-1}(x-iy))\,A^{-1/2}.$$

We now consider the eigenvalues $-d_{i,i}(x)$, $i=1, ..., N$, of $H_{A,B}(x)$, that is, the entries on the diagonal of the diagonal matrix $-D(x)$. It is clear that $-d_{i,i}(x)=a_{i,i}^2(x)-4$ where $a_{i,i}(x)$ is an eigenvalue of $A^{-1/2}(B-xI)\,A^{-1/2}$; if $d_{i,i}(x)<0$, that is, if $a_{i,i}(x)\notin[-2,2]$, there exists a complex neighbourhood of x such that for z in that neighbourhood the real part of $a_{i,i}(z)$ is not in $[-2,2]$, and so according to our choice for the square root, we have

$$\lim_{y\to 0^+}(\sqrt{-d_{i,i}(x+iy)}-\sqrt{-d_{i,i}(x-iy)})=0.$$

In a similar way, if $d_{i,i}(x)>0$, that is, if $a_{i,i}(x)\in(-2,2)$, there exists a complex neighbourhood of x such that for z in that neighbourhood the real part of $a_{i,i}(z)$ is in $(-2,2)$. Taking into account that $a_{i,i}(\bar{z})=\overline{a_{i,i}(z)}$ and that the sign of $\mathscr{I}(a_{i,i}(z))$ is equal to $-\text{sign}(\mathscr{I}(z))$ (because A is definite positive and B is hermitian), we have that

$$\lim_{y\to 0^+}(\sqrt{-d_{i,i}(x+iy)}-\sqrt{-d_{i,i}(x-iy)})=2i\sqrt{d_{i,i}(x)}.$$

Finally, if $d_{i,i}(x)=0$, that is if $a_{i,i}(x)=\pm 2$, we have

$$\lim_{y\to 0^+}(\sqrt{-d_{i,i}(x+iy)}-\sqrt{-d_{i,i}(x-iy)})=0.$$

Hence, it is clear that

$$-\tfrac{1}{2}\lim_{y\to 0^+} A^{-1/2}(U(x+iy)[\sqrt{-D(x+iy)}$$
$$-\sqrt{-D(x-iy)}]\, U^{-1}(x+iy))\, A^{-1/2}$$
$$=-iA^{-1/2}U(x)(D^+(x))^{1/2}\, U^{-1}(x)\, A^{-1/2}.$$

To finish the proof, it is enough to notice that

$$(U(x+iy)\sqrt{-D(x-iy)}\, U^{-1}(x+iy))$$
$$-(U(x-iy)\sqrt{-D(x-iy)}\, U^{-1}(x-iy))$$
$$=[U(x+iy)-U(x-iy)]\sqrt{-D(x-iy)}\, U^{-1}(x+iy)$$
$$+U(x-iy)\sqrt{-D(x-iy)}\,[U^{-1}(x+iy)-U^{-1}(x-iy)],$$

and to take into account that $U(z)$ is analytic at x (and hence continuous) and that $D(z)$ is bounded on the neighbourhood of x.

We now prove the properties of the support of $W_{A,B}$. From its definition, it is clear that the Radon–Nikodym derivative of $W_{A,B}$ is not θ at x if and only if there exists i, $1\leqslant i\leqslant N$, for which $d_{i,i}(x)>0$, that is, if and only if $-H_{A,B}(x)$ has at least one nonnegative eigenvalue. Taking into account that $-H_{A,B}(x)=-(A^{-1/2}(B-xI)\,A^{-1/2})^2+4I$, it follows that the Radon–Nikodym derivative of $W_{A,B}$ is not θ at x if and only if $A^{-1/2}(B-xI)\,A^{-1/2}$ has at least one eigenvalue in $(-2,2)$.

But the support of $W_{A,B}$ can also be described in the following way: consider the polynomials $\Delta_m(H_{A,B}(z))$, $m=1,\ldots,N$, where $\Delta_m(T)$ denotes the principal minor of a matrix T corresponding to its m first rows and columns. Then, we have that

$$\operatorname{supp}(W_{A,B})=\overline{\operatorname{int}(\{x\in\mathbb{R}\colon \text{there exists } m,\ 1\leqslant m\leqslant N \text{ such that } \Delta_m(H_{A,B}(z))\leqslant 0\})}. \tag{3.2}$$

Indeed, the continuity of the eigenvalues gives that the support of $W_{A,B}$ can not have isolated points, hence, the characterization of positive definiteness gives the above description of the support of $W_{A,B}$.

Now, for each m, $1\leqslant m\leqslant N$, the polynomial $\Delta_m(H_{A,B}(z))$ has the form

$$\Delta_m(H_{A,B}(z))=\Delta_m(A^{-2})\,z^{2m}+\text{terms of lower degree}.$$

Since $\Delta_m(A^{-2})>0$, we deduce that the support of $W_{A,B}$ is a union of at most $(N(N+1))/2$ bounded nondegenerate intervals, whose extreme points satisfy $\Delta_m(H_{A,B}(z))=0$ for some m, $1\leqslant m\leqslant N$.

Actually, we can prove more about the support of $W_{A,B}$. To do that, we consider the set of real numbers in the numerical range of the matrix polynomial $H_{A,B}(x)$ (see [GLR, p. 276]), that is, the set

$$\mathbb{R} \cap NR(H_{A,B}) = \{x \in \mathbb{R} \colon \text{there exists } u \in \mathbb{C}^N \backslash \{\theta\} : uH_{A,B}(x)\, u^* = 0\}.$$

It is clear that $\mathbb{R} \cap NR(H_{A,B}) = A_1 \cup A_2 \cup A_3$, where

$$A_1 = \{x \in \mathbb{R} \colon H_{A,B}(x) \text{ is positive semidefinite}\}$$

$$A_2 = \{x \in \mathbb{R} \colon H_{A,B}(x) \text{ is negative semidefinite}\}$$

$$A_3 = \{x \in \mathbb{R} \colon H_{A,B}(x) \text{ has two eigenvalues of opposite sign}\}.$$

From the definition of $W_{A,B}$ it follows that

$$A_3 \subset \operatorname{supp}(W_{A,B}) \tag{3.3}$$

Since $A_1 \cup A_2 \subset \{x \colon \det(H_{A,B}(x)) = 0\}$, we have that

$$A_1 \cup A_2 \text{ is a finite set of real numbers.} \tag{3.4}$$

Using Corollary 10.16, p. 277 of [GLR] we see that $\mathbb{R} \cap NR(H_{A,B})$ is a finite union of at most N disjoint closed intervals and isolated points,

$$\left(\bigcup_{i=1}^{k} [\mu_{2i-1}, \mu_{2i}] \right) \cup \left(\bigcup_{j=1}^{m} \{v_j\} \right),$$

where $\mu_1 < \cdots < \mu_{2k}$, $(k \leqslant N)$, $v_i \neq v_j$, $i \neq j$, are some roots of the scalar polynomial $\det(H_{A,B}(x))$. From (3.3) and (3.4) it follows that

$$\left(\bigcup_{i=1}^{k} [\mu_{2i-1}, \mu_{2i}] \right) \subset \operatorname{supp}(W_{A,B}).$$

We now take an interval $[\alpha, \beta]$, $\alpha < \beta$, for which there exists m, $1 \leqslant m \leqslant N$, such that

$$\Delta_m(H_{A,B}(\alpha)) = \Delta_m(H_{A,B}(\beta)) = 0$$

and (3.5)

$$\Delta_m(H_{A,B}(x)) < 0, \qquad \text{for} \quad x \in (\alpha, \beta).$$

From (3.5) it follows that $H_{A,B}(\alpha)$, $H_{A,B}(\beta)$ are neither positive definite nor negative definite, and so there exists nonnull vectors u_1, u_2 such that $u_1 H_{A,B}(\alpha)\, u_1^* = u_2 H_{A,B}(\beta)\, u_2^* = \theta$. In other words

$$\alpha, \beta \in \mathbb{R} \cap NR(H_{A,B}) = \left(\bigcup_{i=1}^{k} [\mu_{2i-1}, \mu_{2i}] \right) \cup \left(\bigcup_{j=1}^{m} \{v_j\} \right).$$

From this we deduce that

$$[\alpha, \beta] \cup \left(\bigcup_{i=1}^{k} [\mu_{2i-1}, \mu_{2i}] \right)$$

is again a disjoint union of at most N closed nondegenerate intervals whose extreme points are some zeros of the scalar polynomial $\det(H_{A,B}(x))$. Since $[\alpha, \beta] \subset \operatorname{supp}(W_{A,B})$, we have again that

$$[\alpha, \beta] \cup \left(\bigcup_{i=1}^{k} [\mu_{2i-1}, \mu_{2i}] \right) \subset \operatorname{supp}(W_{A,B}).$$

Since this is true for all the intervals satisfying (3.5), (3.2) implies that actually $\operatorname{supp}(W_{A,B})$ is a disjoint union of at most N bounded nondegenerate intervals whose extreme points are some zeros of the scalar polynomial $\det(H_{A,B}(x))$.

We finally prove the continuity of the Radon–Nikodym derivative of $W_{A,B}$. According to [GLR, p. 394], we can choose matrix functions $U(x)$, $D(x)$ which are analytic for x real and such that $-H_{A,B}(x) = U(x)\, D(x)\, U^*(x)$. The result follows because the definition of $(D^+)^{1/2}$ clearly give a continuous matrix function. ∎

We now give some examples:

Example 1. We again take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The system of eigenvalues and eigenvectors for $H_{A,B}(x) = A^{-1/2}(B - xI)\, A^{-1}(B - xI)\, A^{-1/2} - 4I$ (see the first example of the last section) shows that one of the eigenvalues of $H_{A,B}(x)$ is always positive, and the other is negative for $-\frac{5}{2} \leqslant x \leqslant \frac{5}{2}$; hence

$$(D^+(x))^{1/2} = \begin{pmatrix} \dfrac{\sqrt{5\sqrt{16x^2 + 9x^4} - 17x^2}}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

From this we get

$$dW_{A,B}(x) = \frac{1}{2\pi} \frac{\sqrt{5\sqrt{16x^2+9x^4}-17x^2}}{\sqrt{16+9x^2}} \chi_{[-5/2,\,5/2]}(x) \times \begin{pmatrix} \dfrac{3|x|+\sqrt{16+9x^2}}{2\sqrt{2}} & 2\sqrt{2}\,\mathrm{sign}\,(x) \\ 2\sqrt{2}\,\mathrm{sign}(x) & \dfrac{-6\,|x|+2\sqrt{16+9x^2}}{\sqrt{2}} \end{pmatrix} dx,$$

where

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ -1, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Example 2. We take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This second example is not covered by Theorem 3.1, because A is not positive definite. From the expression given in the previous section (Example 3) for $F_{A,B}(z)$ we find, by applying the inversion formula for the Hilbert transform, that

$$dW_{A,B}(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x^2-2x}-1} \times \left[\begin{pmatrix} 1-x & -1 \\ -1 & 1-x \end{pmatrix} \chi_{[1-\sqrt{5},\,0]}(x) + \begin{pmatrix} x-1 & -1 \\ -1 & x-1 \end{pmatrix} \chi_{[2,\,1+\sqrt{5}]}(x) \right] dx.$$

The entries of $W_{A,B}$ are clearly absolutely continuous with respect to the Lebesgue measure, but in this case, their Radon–Nikodym derivatives are not continuous (even not bounded) at $x = 0$ and $x = 2$.

The knowledge of the support of $W_{A,B}$ when A is positive definite allows us to characterize the support of the matrices of measures in the matrix

Nevai's class $M(A, B)$, when A is positive definite. This characterization completes the Blumenthal's theorem for orthogonal matrix polynomials studied in Section 3 of [DL].

Corollary 3.2. *Assume that W is a positive definite matrix of measures in the Nevai class $M(A, B)$, where A is positive definite. Then the support of W is a finite union of at most N disjoint bounded nondegenerate intervals whose extreme points are some roots of the scalar polynomial*

$$\det(A^{-1/2}(B-zI)\,A^{-1}(B-zI)\,A^{-1/2}-4I),$$

and possibly, some sequences of real numbers outside this union which tend to the extreme points of the intervals.

Proof. Consider the orthonormal matrix polynomials $(P_n)_n$ which satisfy the three-term recurrence formula (1.1), where the matrix sequences $(A_n)_n$ and $(B_n)_n$ tend to A and B, respectively. We now form the N-Jacobi matrix associated to this sequence of matrix polynomials, that is,

$$J=\begin{pmatrix} B_0 & A_1 & 0 & 0 & \cdots \\ A_1^* & B_1 & A_2 & 0 & \cdots \\ 0 & A_2^* & B_2 & A_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since $(A_n)_n$ and $(B_n)_n$ converge, the operator associated to the matrix J in the Hilbert space ℓ^2 is bounded, hence from Section 3 of [D3] it follows that W is the unique orthogonalizing matrix of measures for $(P_n)_n$ and that the support of W coincides with the spectrum of J.

Consider now the N-Jacobi matrix associated to the Chebyshev polynomials of the second kind $(U_n^{A,B})_n$, that is (take into account that A is hermitian),

$$\tilde{J}=\begin{pmatrix} B & A & 0 & 0 & \cdots \\ A & B & A & 0 & \cdots \\ 0 & A & B & A & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since $(A_n)_n$ and $(B_n)_n$ tend to A and B, respectively, we deduce that the operator defined by $J-\tilde{J}$ is compact, and then, by a well-known theorem by H. Weyl, J and $\tilde{J}$ have the same essential spectrum. Again by using Section 3 of [D3] we conclude that the spectrum of $\tilde{J}$ is the support of $W_{A,B}$, and the theorem is proved. ∎

When A is positive definite, it is possible to give an expression of the Chebyshev matrix polynomials as matrix functions of Chebyshev scalar polynomials. Indeed, we show that

$$U_n^{A,B}(t) = A^{-1/2} u_n \left(\frac{A^{-1/2}(tI - B)\, A^{-1/2}}{2} \right) A^{1/2}, \tag{3.6}$$

where $(u_n)_n$ is the sequence of Chebyshev polynomials of the second kind. The expression $u_n((A^{-1/2}(tI-B)\, A^{-1/2})/2)$ means the value of the polynomial u_n at the function $((A^{-1/2}(tI-B)\, A^{-1/2})/2)$ defined in the usual way.

Indeed, consider the three-term recurrence relation for the Chebyshev polynomials of the second kind:

$$tu_n(t) = \tfrac{1}{2} u_{n+1}(t) + \tfrac{1}{2} u_{n-1}(t), \qquad u_0(t) = 1. \tag{3.7}$$

Setting $R_0(t) = 1$ and

$$R_n(t) = u_n \left(\frac{A^{-1/2}(tI - B)\, A^{-1/2}}{2} \right), \qquad n \geqslant 1,$$

we get from (3.7) that

$$tA^{-1/2} R_n(t) = A^{1/2} R_{n+1}(t) + BA^{-1/2} R_n(t) + A^{1/2} R_{n-1}(t). \tag{3.8}$$

Taking into account that $U_0^{A,B}(t) = I$, (3.5) follows from (3.7).

4. THE DEGENERATE CASE

In this section we study the case when the limit matrix A is singular.

We start by proving Theorem 1.2 which guarantees the existence of ratio asymptotics also in this case.

Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1 but using the matrix polynomials $R_n(t) = t^n I$ instead of the sequence $(U_n^{A,B})_n$. In this case, we want to guarantee the existence of $\lim_n \int R_l(t)\, d\mu_n(t)$, for $l \geqslant 0$, where the matrices of measures μ_n are defined as in (2.1), although here we are not able to find an explicit expression for these limits as we did in the proof of Theorem 1.1. The reason for this difficulty in finding the value of these limits (which would determine the matrix of measures ν of Theorem 1.2) was given in the Introduction. In Theorems 4.1 and 4.2 of this section we find these limits for some particular cases. We will use a quite different auxiliary sequence of polynomials $(R_l)_l$ in each case, which will illustrate the difficulty of finding the value of these limits in the general case.

As in (2.6), we write again

$$R_l(t)\, P_{n-1}(t) = S_{l,n}(t)\, P_n(t) + \sum_{i=1}^{n} \Delta_{i,l,n} P_{n-i}(t)$$

and find that

$$\int R_l^{A,B}(t)\, d\mu_n(t) = \int \sum_{i=1}^{n} \Delta_{i,l,n} P_{n-i}(t)\, dW(t)\, P_{n-1}^*(t) = \Delta_{1,l,n}.$$

Hence, it is enough to guarantee the existence of $\lim_{n\to\infty} \Delta_{k,l,n}$, for $k, l \geqslant 0$.

We proceed by induction on l. For $l=0$ the result is clear since $\Delta_{1,0,n} = I$ and $\Delta_{i,0,n} = \theta$ for $i \neq 1$.

The definition of the matrix polynomials $(R_l)_l$ and the three-term recurrence formula for $(P_n)_n$ give that

$$\Delta_{k,l+1,n} = \Delta_{k,l,n} B_{n-k} + \Delta_{k-1,l,n} A_{n-k+1}^* + \Delta_{k+1,l,n} A_{n-k}.$$

Using the induction hypothesis and $\lim_n A_n = A$, $\lim_n B_n = B$, it is easy to conclude that $\lim_{n\to\infty} \Delta_{k,l,n}$ exists for $k, l \geqslant 0$.

We set $\Delta_{1,l}$ for $\lim_n \Delta_{1,l,n}$, $l \geqslant 0$.

We now take any limit point ν of (μ_n) (there exists such a limit from Banach and Alaoglu's theorem because μ_n is a positive definite matrix of measures and $\int d\mu_n = I$, $n \in \mathbb{N}$), and proceeding as in the proof of Theorem 1.1, we deduce that

$$\int R_l(t)\, d\nu(t) = \Delta_{1,l}, \qquad l \geqslant 0.$$

But ν has compact support (see Lemma 2.1), and since the matrix polynomials $(R_l)_l$ form a basis of the space of matrix polynomials, ν is determined by $\int R_l(t)\, d\nu(t)$. Actually we have proved that $(\mu_n)_n$ has only one limit point ν, and hence, this ν is the limit of $(\mu_n)_n$. Hence, it follows that

$$\lim_n \int \frac{d\mu_n(t)}{z-t} = \int \frac{\nu(t)}{z-t}.$$

We now conclude again as in the proof of Theorem 1.1.

Putting $F_{A,B}(z) = \int (d\nu(t)/(z-t))$, we get from the three-term recurrence relation for the matrix polynomials $(P_n)_n$ that

$$zI = F_{A,B}^{-1}(z) + B + A^* F_{A,B}(z)\, A,$$

and hence

$$A^*F_{A,B}(z)\,AF_{A,B}(z)+(B-zI)\,F_{A,B}(z)+I=\theta. \quad \blacksquare$$

We now prove that the matrix of measures ν which appears in Theorem 1.2 is always degenerate. More precisely, we show that $\int (tI-B)\,d\nu(t)(tI-B)$ is singular, and therefore ν can not have a sequence of orthogonal matrix polynomials.

To do that we compute the first three moments of the matrices of measures μ_n defined by (2.1). The quadrature formula for the orthonormal matrix polynomials $(P_n)_n$ gives that

$$\int d\mu_n(t)=I, \qquad n\geqslant 0.$$

By using again this quadrature formula we have

$$\int t\,d\mu_n(t)=\int tP_{n-1}(t)\,dW(t)\,P^*_{n-1}(t),$$

where W is the matrix weight for $(P_n)_n$. The three-term recurrence formula for these polynomials gives that

$$\int t\,d\mu_n(t)=B_{n-1}, \qquad n\geqslant 0.$$

By using again this three-term recurrence relation we have:

$$\int t^2\,d\mu_n(t)=\sum_{k=1}^{m} x_{n,k}(A_nP_n(x_{nk})+B_{n-1}P_{n-1}(x_{nk})$$
$$+A^*_{n-1}P_{n-2}(x_{nk}))\,\Gamma_{nk}P^*_{n-1}(x_{nk}).$$

The definition of Γ_{nk} and Theorem 2.3(3), p. 1184 of [D4] show that $P_n(x_{nk})\,\Gamma_{nk}=\theta$, and then the quadrature formula for the orthonormal matrix polynomials $(P_n)_n$ gives that

$$\int t^2\,d\mu_n(t)=B_{n-1}\int tP_{n-1}(t)\,dW(t)\,P^*_{n-1}(t)$$
$$+A^*_{n-1}\int tP_{n-2}(t)\,dW(t)\,P^*_{n-1}(t).$$

Using again the three-term recurrence relation we find

$$\int t^2 \, d\mu_n(t) = B_{n-1} B_{n-1} + A_{n-1}^* A_{n-1}.$$

From this computation we get

$$\int (tI - B) \, d\mu_n(tI - B) = B_{n-1} B_{n-1} - BB_{n-1} - B_{n-1} B + BB + A_{n-1}^* A_{n-1}.$$

Taking the limit as $n \to \infty$ we finally have that

$$\int (tI - B) \, d\nu(tI - B) = A^* A,$$

which is singular because A is so.

In the scalar case a positive measure μ is degenerate, i.e., μ has moments of every order but does not have a sequence of orthogonal polynomias, if and only if μ has finite support, or in other words, μ is a finite combination of Dirac deltas. In the matrix case, there are more possibilities for a positive definite matrix of measures W to be degenerate (see [D3, p. 96]). The rest of this section is devoted to some examples

EXAMPLE 1. Assume

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad a \neq 0, \quad B = \begin{pmatrix} b & c \\ c & d \end{pmatrix}, \quad c \neq 0. \tag{4.1}$$

As we wrote in the Introduction this is a very interesting case because it corresponds with scalar orthonormal polynomials with periodic recurrence coefficients (period 2) defined by

$$tp_n(t) = a_{n+1} p_{n+1}(t) + b_n p_n(t) + a_n p_{n-1}(t), \qquad n \geqslant 1, \tag{4.2}$$

where $p_0(t) = 1$, $p_{-1}(t) = 0$, and

$$a_{2n} = a, \qquad a_{2n+1} = c,$$

$$b_{2n} = b, \qquad b_{2n+1} = d.$$

In this case, we can identify the matrix of measures which appears in Theorem 1.2:

THEOREM 4.1. *Assume that A and B is given by* (4.1). *Then the matrix of measures* v *which appears in the Theorem* 1.2 *is given by*

$$v = 1\begin{pmatrix} 1 & \dfrac{t-b}{c} \\ \dfrac{t-b}{c} & \left(\dfrac{t-b}{c}\right)^2 \end{pmatrix}\mu,$$

where μ *is the positive measure with respect to which the sequence of scalar polynomials defined by* (4.2) *is orthonormal.*

Proof. To simplify we assume that

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}.$$

We consider the matrix J^2, where J is defined by (1.9),

$$J^2 = \begin{pmatrix} \tilde{B}_0 & \tilde{A}^* & \theta & \theta & \cdots \\ \tilde{A} & \tilde{B} & \tilde{A}^* & \theta & \cdots \\ \theta & \tilde{A} & \tilde{B} & \tilde{A}^* & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\tilde{B}_0 = B^2 + A^*A$$

$$\tilde{B} = B^2 + AA^* + A^*A$$

$$\tilde{A} = AB + BA;$$

the matrix J^2 is again block three diagonal because $A^2 = \theta$. According to the definition of A and B, we have $\tilde{A} = \left(\begin{smallmatrix} a & ad \\ 0 & a \end{smallmatrix}\right)$, from which we find that $\tilde{A}$ is nonsingular. Hence, we can define the matrix polynomials R_l, $l \geqslant 0$, by

$$\begin{aligned} R_0(t) &= I, \\ t^2 R_0 &= \tilde{A}^* R_2(t) + \tilde{B}_0 R_0(t), \\ t^2 R_{2l} &= \tilde{A}^* R_{2l+2}(t) + \tilde{B} R_{2l}(t) + \tilde{A} R_{2l-2}(t), \qquad && l \geqslant 1, \\ R_{2l+1}(t) &= (t - B)\, R_{2l}(t), && l \geqslant 0. \end{aligned}$$

We proceed again as in the proof of Theorem 1.1 but using the sequence $(R_l)_l$ instead of $(U_l^{A,B})_l$. Writing

$$R_{2l}(t)\,P_{n-1}(t)=S_{l,n}(t)\,P_n(t)+\sum_{k=1}^{n}\Delta_{k,l,n}P_{n-k}(t), \tag{4.3}$$

we prove that

$$\lim_n \Delta_{k,l,n}=\begin{cases} I, & \text{if}\quad k=l+1\\ \theta, & \text{otherwise.}\end{cases} \tag{4.4}$$

We proceed by induction on l. The case $l=0$ is trivial. For $l=1$, a straightforward computation gives that

$$\Delta_{1,1,n}=\tilde{A}^{*-1}[(B_{n-1}B_{n-1}+A_{n-1}^*A_{n-1})-\tilde{B}_0],$$

$$\Delta_{2,1,n}=\tilde{A}^{*-1}(B_{n-1}A_{n-1}^*+A_{n-1}^*B_{n-2}),$$

$$\Delta_{3,1,n}=\tilde{A}^{*-1}(A_{n-1}^*A_{n-2}^*).$$

Hence $\lim_n \Delta_{1,1,n}=\theta$, $\lim_n \Delta_{2,1,n}=I$ and $\lim_n \Delta_{3,1,n}=\theta$ (the latter because $A^2=\theta$).

By using the definition of R_{2l} and the three-term recurrence relation for $(P_n)_n$ we get for $k\geqslant 2$ that

$$\begin{aligned}\Delta_{k,l+1,n}=\tilde{A}^{*-1}(&\Delta_{k+2,l,n}[A_{n-k-1}A_{n-k}]\\ &+\Delta_{k+1,l,n}[B_{n-k-1}A_{n-k}+A_{n-k}B_{n-k}]\\ &+\Delta_{k,l,n}[A_{n-k+1}A_{n-k+1}^*+B_{n-k}^2+A_{n-k}^*A_{n-k}]\\ &+\Delta_{k-1,l,n}[B_{n-k+1}A_{n-k+1}^*+A_{n-k+1}^*B_{n-k}]\\ &+\Delta_{k-2,l,n}[A_{n-k+2}^*A_{n-k+1}^*])\\ &-\tilde{A}^{*-1}\tilde{B}\,\Delta_{k,l,n}-\tilde{A}^{*-1}\tilde{A}\,\Delta_{k,l-1,n}\end{aligned}$$

and for $k=1$ that

$$\begin{aligned}\Delta_{1,l+1,n}=\tilde{A}^{*-1}(&\Delta_{3,l,n}[A_{n-2}A_{n-1}]+\Delta_{2,l,n}[B_{n-2}A_{n-1}+A_{n-1}B_{n-1}]\\ &+\Delta_{1,l,n}[B_{n-1}^2+A_{n-1}^*A_{n-1}]).\end{aligned}$$

The induction hypothesis, the fact that $A^2=A^{*2}=\theta$ and an easy computation show that (4.4) holds.

Since $\int R_{2l}(t)\,dv(t)=\lim_n \Delta_{1,l,n}$, we have that

$$\int R_{2l}(t)\,dv(t)=\begin{cases} I, & \text{if } \ l=0 \\ \theta, & l\geqslant 1. \end{cases}$$

We now show that

$$\int R_{2l+1}(t)\,dv(t)=\begin{cases} A, & l=1 \\ \theta, & l\neq 1. \end{cases} \tag{4.5}$$

Using (4.3) and the quadrature formula for the orthonormal polynomials $(P_n)_n$, it follows that

$$\begin{aligned}\int R_{2l+1}(t)\,dv(t)&=\int (t-B)\,R_{2l}(t)\,dv(t)\\ &=\lim_n\,(\Delta_{1,l,n}B_{n-1}+\Delta_{2,l,n}A_{n-1}-B\Delta_{1,l,n}),\end{aligned}$$

and hence, (4.5) follows from (4.4).

We now find an expression of the matrix polynomials $(R_l)_l$ in terms of the scalar polynomials $(p_n)_n$ defined by (4.2). Indeed, from the definition of the sequence of polynomials R_l, it follows that R_{2l} is even for all $l\geqslant 0$, hence, by setting $T_l(t)=R_{2l}(\sqrt{t})$, we have defined a sequence of polynomials $(T_l)_l$ which satisfies the three-term recurrence formula

$$\begin{gathered} T_0(t)=I, \qquad tT_0=\tilde{A}^*T_1(t)+\tilde{B}_0T_0(t),\\ tT_l=\tilde{A}^*T_{l+1}(t)+\tilde{B}T_l(t)+\tilde{A}T_{l-1}(t), \qquad l\geqslant 1. \end{gathered}$$

If we iterate the three term recurrence relation given by (4.2), we find a five term recurrence relation for the polynomials $(p_n)_n$ of the form

$$t^2p_n(t)=\alpha_{n+2}p_{n+2}(t)+\beta_{n+1}p_{n+1}(t)+\gamma_np_n(t)+\beta_np_{n-1}(t)+\alpha_np_{n-2}(t). \tag{4.6}$$

If we consider the 5-Jacobi matrix $\tilde{J}$ associated to this five term recurrence relation, that is, the 5-banded infinite hermitian matrix defined by putting the sequences $(\alpha_n)_n$, $(\beta_n)_n$, $(\gamma_n)_n$ which appear in the recurrence relation (4.6) on the diagonals of the matrix $\tilde{J}$, we find that this matrix $\tilde{J}$ just coincides with J^2. Hence by using the relationship between scalar polynomials satisfying a five term recurrence relation as (4.6) and 2×2 orthonormal matrix polynomials given in Section 2 of [DV], we deduce that

$$R_{2l}(t)=T_l(t^2)=\begin{pmatrix} p_{2l,E}(t) & \dfrac{p_{2l,O}(t)}{t} \\ p_{2l+1,E}(t) & \dfrac{p_{2l+1,O}(t)}{t} \end{pmatrix},$$

where $p_{n,E}$, $p_{n,O}$ denote the even and odd parts of the polynomial p_n, that is,

$$p_{n,E}(t)=\frac{p_n(t)+p_n(-t)}{2}, \qquad p_{n,O}(t)=\frac{p_n(t)-p_n(-t)}{2}.$$

Consider now the matrix of measures $\tilde{v}$ given by

$$\tilde{v}=\begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}\mu$$

where μ is the positive measure with respect to which the sequence of scalar polynomials defined by (4.2) is orthonormal. Then, taking into account that $p_n=p_{n,E}+p_{n,O}$, we have that

$$\int R_{2l}(t)\,d\tilde{v}(t)=\int \begin{pmatrix} p_{2l}(t) & tp_{2l}(t) \\ p_{2l+1}(t) & tp_{2l+1}(t) \end{pmatrix} d\mu(t),$$

$$\int R_{2l+1}(t)\,d\tilde{v}(t)=\int \begin{pmatrix} tp_{2l}(t) & t^2p_{2l}(t) \\ tp_{2l+1}(t) & t^2p_{2l+1}(t) \end{pmatrix} d\mu(t)$$

$$-B\int \begin{pmatrix} p_{2l}(t) & tp_{2l}(t) \\ p_{2l+1}(t) & tp_{2l+1}(t) \end{pmatrix} d\mu(t).$$

The orthonormality of $(p_n)_n$ with respect to μ and the recurrence formula (4.2) gives that for $l\geqslant 3$

$$\int R_l(t)\,d\tilde{v}(t)=\theta.$$

Taking into account the form of A and B, we deduce that

$$\int R_0(t)\,d\tilde{v}(t)=I, \qquad \int R_1(t)\,d\tilde{v}(t)=\theta,$$

$$\int R_2(t)\,d\tilde{v}(t)=\theta, \qquad \int R_3(t)\,d\tilde{v}(t)=A.$$

Since the sequence $(R_l)_l$ forms a basis of the linear space of polynomials, any matrix of measures W with compact support is determined by $\int R_l(t)\,dW(t)$ and hence we deduce that $v=\tilde{v}$, and the theorem is proved. ∎

In a similar way the general case of orthogonal polynomials with periodic recurrence coefficients of period N can be studied. This case appears when A and B have the form given by (1.10).

EXAMPLE 2. Another possibility for the degenerateness of the matrix of measures v appears when v is equal to a matrix whose entries are Dirac deltas, in some subspace of $\mathbb{C}^N$. We now give a condition on the matrices A and B which implies the occurrence of this case

THEOREM 4.2. *Assume that there exists* $u \in \operatorname{Ker}(A^*)\setminus\{\theta\}$ *such that* $uB^n \in \operatorname{Ker}(A^*)$ *for all* $n \geqslant 0$. *Set* V *for the linear span of* uB^n, $n \geqslant 0$. *Consider the diagonal form of* B (*let us recall that* B *is hermitian*),

$$B=U^*\begin{pmatrix} d_1 & 0 & \cdots & 0\\ 0 & d_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & d_N\end{pmatrix}U,$$

where $U^*U=I$ *and* $d_1, ..., d_N$ *are real numbers. Then, the matrix of measure* v *which appears in Theorem* 1.2 *is such that* $vv=v\tilde{v}$ *for* $v\in V$, *where* $\tilde{v}$ *is the matrix of measures defined by*

$$\tilde{v}=U^*\begin{pmatrix} \delta_{d_1} & 0 & \cdots & 0\\ 0 & \delta_{d_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \delta_{d_N}\end{pmatrix}U.$$

Before going to the proof, we remark that the assumption in this theorem automatically holds if some of the eigenvectors of B belong to $\operatorname{Ker}(A^*)$.

Proof. Consider the matrix polynomials $R_l(t)=(tI-B)^l$, $l\geqslant 0$, and write

$$R_l(t)\,P_{n-1}(t)=S_{l,n}(t)\,P_n(t)+\sum_{k=1}^{n}\Delta_{k,l,n}P_{n-k}(t),$$

as in the proof of Theorem 1.1. Then, for $v \in V$ we prove that

$$\lim_{n\to\infty} v\Delta_{k,l,n} = \begin{cases} v, & l=0, \\ \theta, & l \geqslant 1. \end{cases}$$

Indeed, using the definition of the polynomials R_l and the three-term recurrence relation for $(P_n)_n$ we find the formula

$$\Delta_{k,l+1,n} = \Delta_{k,l,n} B_{n-k} + \Delta_{k-1,l,n} A^*_{n-k+1} + \Delta_{k+1,l,n} A_{n-k} - B\,\Delta_{k,l,n}.$$

It is now enough to proceed by induction, taking into account that $V \subset \operatorname{Ker}(A^*)$ and $vB \in V$, for all $v \in V$.

As in the proof of Theorem 1.1, it follows that

$$\int R_l(t)\, d\mu_n(t) = \Delta_{1,l,n},$$

where μ_n are the matrix of measures defined by (2.1). Since ν is the limit of μ_n, we deduce that for $v \in V$,

$$v \int R_l(t)\, d\nu(t) = \begin{cases} v, & l=0, \\ \theta, & l \geqslant 1. \end{cases}$$

But, from the definition of the matrix of measures $\tilde{\nu}$, it is clear that for any $v \in \mathbb{C}^N$,

$$v \int R_l(t)\, d\tilde{\nu}(t) = \begin{cases} v, & l=0, \\ \theta, & l \geqslant 1. \end{cases}$$

Since $(R_l)_l$ is a basis of the space of matrix polynomials, and ν, $\tilde{\nu}$ have compact support, we get that $v\nu = v\tilde{\nu}$, for $v \in V$. ∎

To illustrate this case, we give the following example:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Here, $u = (0, 1) \in \operatorname{Ker}(A^*)$ and $uB = 2u$. Solving the equation for $\int (d\nu(t)/(z-t))$, we have that

$$\int \frac{d\nu(t)}{z-t} = \begin{pmatrix} \dfrac{z-2}{z^2-3z+1} & 0 \\ 0 & \dfrac{1}{z-2} \end{pmatrix},$$

from which we find

$$v = \begin{pmatrix} \dfrac{5+\sqrt{5}}{10}\,\delta_{(3-\sqrt{5})/2} + \dfrac{5-\sqrt{5}}{10}\,\delta_{(3+\sqrt{5})/2} & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

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